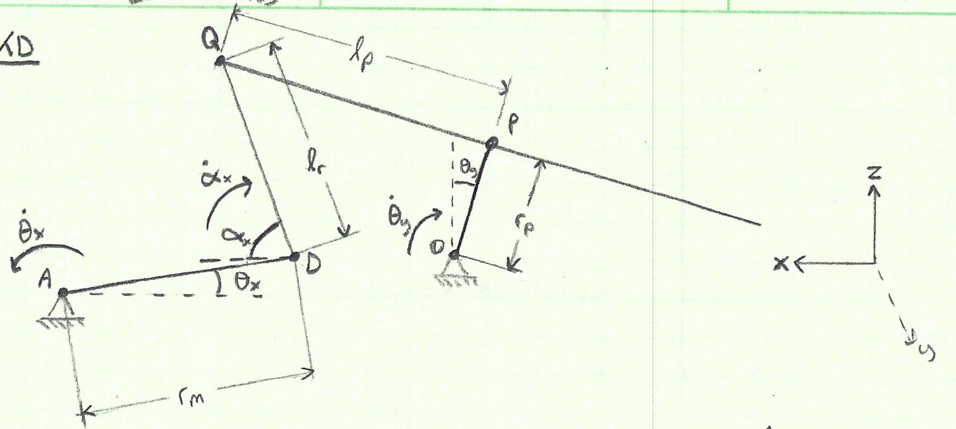
1 KD



ASSUMPTIONS

1) ROTATIONS OF PLATFORM ARE INDEPENDENT
2) RIGID BODIES
3) 2D KINEMATICS (MOTOR LIES IN XZ-PLANE)
4) @ $\theta_x = 0$, $\theta_y = 0$, $\alpha_x = \alpha_0$

ANALYSIS

FIRST FIND RELATIVE MOTIONS

$$\vec{V_Q} = \vec{V_D} + \vec{V_{Q/D}}$$

$$\vec{\omega_Q} \times \vec{r_Q} = \vec{\omega_D} \times \vec{r_D} + \vec{\omega_{Q/D}} \times \vec{r_{Q/D}}$$

$$\dot{\theta}_y \hat{j} \times \left[(r_p \cos\theta_y + l_p \sin\theta_y)\hat{k} + (l_p \cos\theta_y - r_p \sin\theta_y)\hat{i} \right] =$$

$$-\dot{\theta}_x \hat{j} \times r_m (\sin\theta_x \hat{k} - \cos\theta_x \hat{i}) + \dot{\alpha}_x \hat{j} \times l_r (\sin\alpha_x \hat{k} + \cos\alpha_x \hat{i})$$

$$\dot{\theta}_y \left[(r_p \cos\theta_y + l_p \sin\theta_y)\hat{i} - (l_p \cos\theta_y - r_p \sin\theta_y)\hat{k} \right] =$$

$$\left[\dot{\alpha}_x l_r \sin\alpha_x - \dot{\theta}_x r_m \sin\theta_x \right]\hat{i} - \left[\dot{\theta}_x r_m \cos\theta_x + \dot{\alpha}_x l_r \cos\alpha_x \right]\hat{k}$$

$$\hat{i}: \dot{\theta}_y (r_p \cos\theta_y + l_p \sin\theta_y) = \dot{\alpha}_x l_r \sin\alpha_x - \dot{\theta}_x r_m \sin\theta_x$$

$$\hat{j}: \dot{\theta}_y (r_p \sin\theta_y - l_p \cos\theta_y) = -(\dot{\theta}_x r_m \cos\theta_x + \dot{\alpha}_x l_r \cos\alpha_x)$$

NOW ISOLATE $\dot{\theta}_x$ TERMS FOR BOTH EQNs

$$\hat{i}: \dot{\alpha}_x l_r \sin\alpha_x - \dot{\theta}_y (r_p \cos\theta_y + l_p \sin\theta_y) = \dot{\theta}_x r_m \sin\theta_x$$

$$\hat{j}: \dot{\alpha}_x l_r \cos\alpha_x + \dot{\theta}_y (r_p \sin\theta_y - l_p \cos\theta_y) = -\dot{\theta}_x r_m \cos\theta_x$$

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1 CONT.

NEXT CONVERT TO MATRIX FORM

$$\begin{array}{cc|c} \dot{\alpha}_x & \dot{\theta}_y & \dot{\theta}_x \\ \ell_r \sin\alpha_x & -(r_p \cos\theta_y + \ell_p \sin\theta_y) & r_m \sin\theta_x \\ \ell_r \cos\alpha_x & r_p \sin\theta_y - \ell_p \cos\theta_y & -r_m \cos\theta_x \end{array}$$

NOW SOLVE USING ROW OPERATIONS

$R_1 \leftarrow R_1 / \ell_r \sin\alpha_x$

$R_2 \leftarrow R_2 / \ell_r \cos\alpha_x$

$$\left[\begin{array}{cc|c} 1 & \frac{-(r_p \cos\theta_y + \ell_p \sin\theta_y)}{\ell_r \sin\alpha_x} & \frac{r_m \sin\theta_x}{\ell_r \sin\alpha_x} \\ 1 & \frac{r_p \sin\theta_y - \ell_p \cos\theta_y}{\ell_r \cos\alpha_x} & \frac{-r_m \cos\theta_x}{\ell_r \cos\alpha_x} \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$$\left[\begin{array}{cc|c} 1 & \frac{-(r_p \cos\theta_y + \ell_p \sin\theta_y)}{\ell_r \sin\alpha_x} & \frac{r_m \sin\theta_x}{\ell_r \sin\alpha_x} \\ 0 & \left(\frac{r_p \sin\theta_y - \ell_p \cos\theta_y}{\cancel{\ell_r} \cos\alpha_x}\right) + \left(\frac{r_p \cos\theta_y + \ell_p \sin\theta_y}{\cancel{\ell_r} \sin\alpha_x}\right) & -\left(\frac{r_m \cos\theta_x}{\cancel{\ell_r} \cos\alpha_x} + \frac{r_m \sin\theta_x}{\cancel{\ell_r} \sin\alpha_x}\right) \end{array}\right]$$

SIMPLIFY R_2:

$$-\dot{\theta}_y \left[\sin\alpha_x (r_p \sin\theta_y - \ell_p \cos\theta_y) + \cos\alpha_x (r_p \cos\theta_y + \ell_p \sin\theta_y)\right] = \dot{\theta}_x r_m \left[\sin\alpha_x \cos\theta_x + \sin\theta_x \cos\alpha_x\right]$$

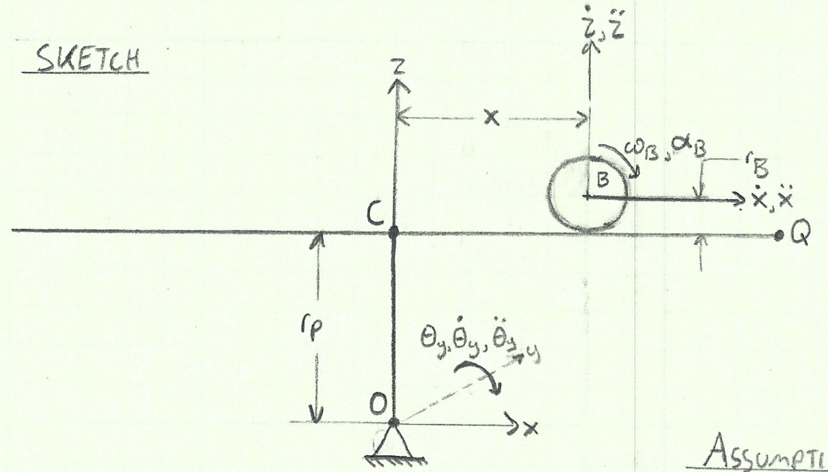
$$-\dot{\theta}_y \left[r_p (\sin\alpha_x \sin\theta_y + \cos\alpha_x \cos\theta_y) + \ell_p (\sin\theta_y \cos\alpha_x - \cos\theta_y \sin\alpha_x)\right] = \dot{\theta}_x r_m \sin(\theta_x + \alpha_x)$$

$$-\dot{\theta}_y \left[r_p \cos(\theta_y - \alpha_x) + \ell_p \sin(\theta_y - \alpha_x)\right] = \dot{\theta}_x r_m \sin(\theta_x + \alpha_x)$$

$$\boxed{\dot{\theta}_y = \dot{\theta}_x \frac{r_m \sin(\alpha_x + \theta_x)}{\ell_p \sin(\alpha_x - \theta_y) - r_p \cos(\alpha_x - \theta_y)}}$$

2

SKETCH



Assumptions
1) Platform rotations are independent
2) Rigid bodies
3) 2D kinematics
4) Ball rolls without slipping

ANALYSIS

$$\vec{V_B} = \vec{V_O} + \vec{\Omega}\times\vec{r_{B/O}} + \vec{V_{B\,rel}} \qquad (\vec{V_O} = 0)$$

$$\dot{x}\hat{i} + \dot{z}\hat{k} = \dot{\theta}_y\hat{j}\times\left[x\hat{i} + (r_p + r_B)\hat{k}\right] + \omega_B\hat{j}\times r_B\hat{k}$$

$$\dot{x}\hat{i} + \dot{z}\hat{k} = -\dot{\theta}_y x\hat{k} + \dot{\theta}_y(r_p + r_B)\hat{i} + \omega_B r_B\hat{i}$$

$\hat{i}$: $\boxed{\dot{x} = \dot{\theta}_y(r_p + r_B) + \omega_B r_B}$

$\hat{k}$: $\dot{z} = -\dot{\theta}_y x$

$$\vec{a_B} = \vec{a_O} + \dot{\vec{\Omega}}\times\vec{r_{B/O}} + \vec{\Omega}\times\vec{\Omega}\times\vec{r_{B/O}} + 2\vec{\Omega}\times\vec{V_{B\,rel}} + \vec{a_{A\,rel}} \qquad (\vec{a_O} = 0)$$

$$\ddot{x}\hat{i} + \ddot{z}\hat{k} = \ddot{\theta}_y\hat{j}\times\left[x\hat{i} + (r_p + r_B)\hat{k}\right] + \dot{\theta}_y\hat{j}\times\left[-\dot{\theta}_y x\hat{k} + \dot{\theta}_y(r_p + r_B)\hat{i}\right]$$
$$+ 2\dot{\theta}_y\hat{j}\times\omega_B r_B\hat{i} + \alpha_B\hat{j}\times r_B\hat{k} + \omega_B\hat{j}\times\omega_B r_B\hat{i}$$

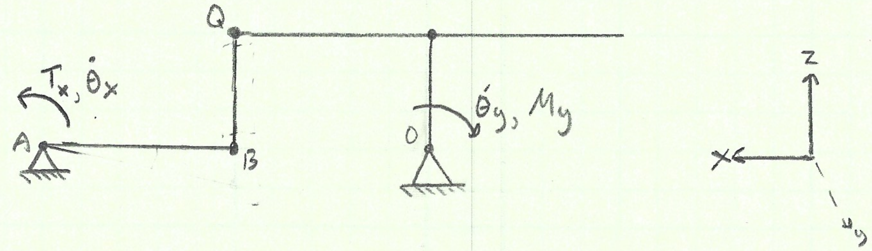
$$\ddot{x}\hat{i} + \ddot{z}\hat{k} = -\ddot{\theta}_y x\hat{k} + \ddot{\theta}_y(r_p + r_B)\hat{i} - \dot{\theta}_y^2 x\hat{i} - \dot{\theta}_y^2(r_p + r_B)\hat{k} - 2\dot{\theta}_y\omega_B r_B\hat{k} + \alpha_B r_B\hat{i} - \omega_B^2 r_B\hat{k}$$

$\hat{i}$: $\boxed{\ddot{x} = \ddot{\theta}_y(r_p + r_B) - \dot{\theta}_y^2 x + \alpha_B r_B}$

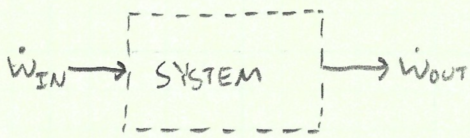
$\hat{k}$: $\ddot{z} = -\ddot{\theta}_y x - \dot{\theta}_y^2(r_p + r_B) - 2\dot{\theta}_y\omega_B r_B - \omega_B^2 r_B$

3

SKETCH



ANALYSIS



$$\dot{W}_{IN} = \dot{W}_{OUT}$$

$$T_x \dot{\theta}_x = -M_y \dot{\theta}_y$$

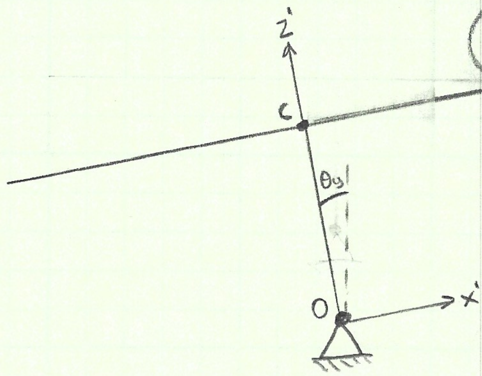
$$M_y = T_x \frac{\dot{\theta}_x}{\dot{\theta}_y}$$

$$M_y = T_x \frac{r_m \sin(\alpha_x + \theta_x)}{l_p \sin(\alpha_x - \theta_y) - r_p \cos(\alpha_x - \theta_y)}$$

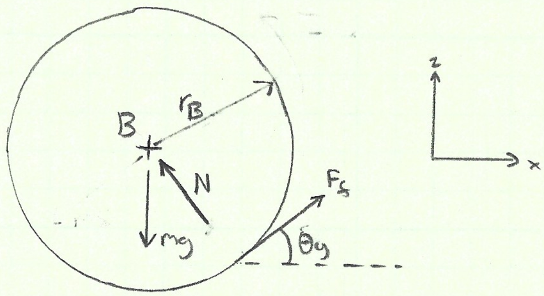
ASSUMPTIONS
1) PLATFORM ROTATIONS ARE INDEPENDENT
2) RIGID BODIES
3) 2D KINEMATICS
4) MASS OF THE BODIES IS NEGLIGIBLE

4

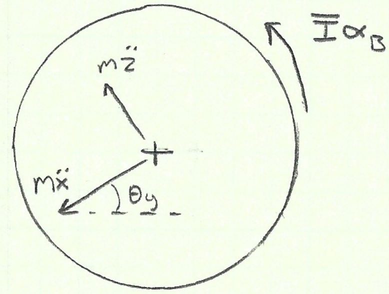
SKETCH



FBD - BALL ONLY



KD - BALL ONLY



$$\Sigma M_B \Big|_{FBD} = \Sigma M_B \Big|_{KD}$$

$$r_B F_f = \bar{I}\alpha_B$$

$$F_f = \frac{\frac{2}{5} m r_B^{\not 2} \alpha_B}{\not r_B}$$

$$F_f = \frac{2}{5} m r_B \alpha_B$$

$$\Sigma F_x \Big|_{FBD} = \Sigma F_x \Big|_{KD}$$

$$-N \sin\theta_y + F_f \cos\theta_y = -m\ddot{x}\cos\theta_y - m\ddot{z}\sin\theta_y$$

$$N = \frac{m\ddot{x} + F_f}{\tan\theta_y} + m\ddot{z}$$

$$\Sigma F_z \Big|_{FBD} = \Sigma F_z \Big|_{KD}$$

$$-mg + N\cos\theta_y + F_f \sin\theta_y = -m\ddot{x}\sin\theta_y + m\ddot{z}\cos\theta_y$$

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4 CONT.

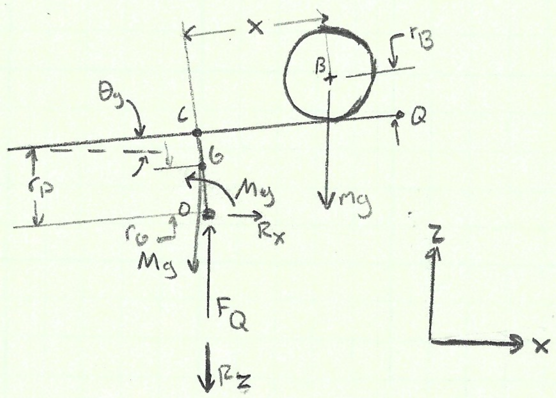
$$-\cancel{m}g + \left(\cancel{m}\ddot{x} + \tfrac{2}{5}\cancel{m}r_B\alpha_B\right)\sin\theta_y + \tfrac{2}{5}\cancel{m}r_B\alpha\sin\theta_y = -\cancel{m}\ddot{x}\sin\theta_y + \cancel{m\ddot{z}\cos\theta_y}$$

$$\cancel{+m\ddot{z}\cos\theta_y}$$

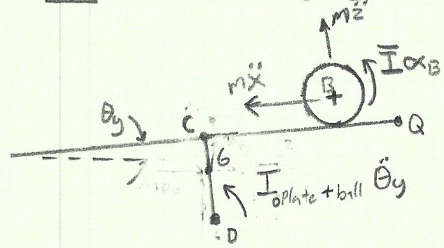
$$\frac{-g}{\sin\theta_y} - \ddot{x} - \frac{4}{5}r_B\alpha_B = \ddot{x}$$

(0)
$$\boxed{\left(\frac{g}{\sin\theta_y} - \frac{4}{5}r_B\alpha_B\right)\frac{1}{2} = \ddot{x}}$$

FBD - PLATE AND BALL



KD - PLATE AND BALL



$$\Sigma M_O\big|_{FBD} = \Sigma M_O\big|_{KD}$$

(1)
$$M_y + r_G M_g \sin\theta_y - x mg\cos\theta_y + (r_p + r_B)mg\sin\theta_y = (r_p + r_B)m\ddot{x} + \bar{I}_B\alpha_B + I_{O\,plate+ball}\ddot{\theta}_y + x m\ddot{z}$$

$$\Sigma F_x\big|_{FBD} = \Sigma F_x\big|_{KD}$$

(2)
$$R_x = -m\ddot{x}\cos\theta_y - m\ddot{z}\sin\theta_y$$

$$\Sigma F_z\big|_{FBD} = \Sigma F_z\big|_{KD}$$

(3)
$$F_Q - R_z - M_g - mg = -m\ddot{x}\sin\theta_y + m\ddot{z}\cos\theta_y$$

*NOTE: $F_Q = \dfrac{M_y}{l_p\cos\theta_y}$

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4 Cont.

SUBSTITUTE KNOWN VALUES INTO EQ ①
- M_y from #3
- α_B from EQ ⑥

$$T_x \frac{r_m \sin(\alpha_x+\theta_x)}{l_p \sin(\alpha_x-\theta_y) - r_p\cos(\alpha_x-\theta_y)} + \left[\left(r_G M + [r_p+r_B]m\right)\sin\theta_y - x m\cos\theta_y\right]g$$

$$= (r_p+r_B)m\ddot{x} + \tfrac{1}{2}m r_B^2\left[\left(\frac{g}{\sin\theta_y} - 2\ddot{x}\right)\frac{1}{r_B}\right] + I_{O,plat+ball}\ddot{\theta}_y$$

$$+ xm\left[-\ddot{\theta}_y x - \dot{\theta}_y^2(r_p+r_B) - 2\dot{\theta}_y\omega_B r_B - \omega_B^2 r_B\right]$$

$$\frac{T_x r_m \sin(\alpha_x+\theta_x)}{l_p\sin(\alpha_x-\theta_y) - r_p\cos(\alpha_x-\theta_y)} + \left[\left(r_G M + [r_p+r_B]m\right)\sin\theta_y - m\left(x\cos\theta_y - \frac{r_B}{2\sin\theta_y}\right)\right]g$$

$$+ xm\left[\dot{\theta}_y^2(r_p+r_B) + 2\dot{\theta}_y\omega_B r_B - \omega_B^2 r_B\right]$$

$$= \ddot{x}\, m(r_p + r_B - r_B) + \ddot{\theta}_y\left(I_{O,plat+ball} - x^2 m\right)$$

SUBSTITUTE ω_B FROM #2

$$\omega_B = \frac{\dot{x} - \dot{\theta}_y(r_p+r_B)}{r_B}$$

$$\frac{T_x r_m \sin(\alpha_x+\theta_x)}{l_p\sin(\alpha_x-\theta_y) - r_p\cos(\alpha_x-\theta_y)} + \left[\left(r_G M + [r_p+r_B]m\right)\sin\theta_y + \left(\frac{r_B}{\sin(2\theta_y)} - x\right)m\cos\theta_y\right]g$$

$$+ xm\left[\dot{\theta}_y^2(r_p+r_B) + 2\dot{\theta}_y\left(\frac{\dot{x} - \dot{\theta}_y(r_p+r_B)}{r_B}\right)r_B - \left(\frac{\dot{x} - \dot{\theta}_y(r_p+r_B)}{r_B}\right)^2 r_B\right]$$

$$= \ddot{x}\, m r_p + \ddot{\theta}_y\left(I_{O,plat+ball} - x^2 m\right)$$

$$\boxed{\begin{aligned}&\frac{T_x r_m \sin(\alpha_x+\theta_x)}{l_p\sin(\alpha_x-\theta_y) - r_p\cos(\alpha_x-\theta_y)} + \left[\left(r_G M + [r_p+r_B]m\right)\sin\theta_y + \left(\frac{r_B}{\sin(2\theta_y)} - x\right)m\cos\theta_y\right]g \\ &- \frac{xm}{r_B}\left[\left(\dot{x} - r_B\dot{\theta}_y\right)^2 + \dot{\theta}_y^2\left[(r_p+r_B)(r_B-1) - r_B^2\right]\right] = \ddot{x}\, m r_p + \ddot{\theta}_y\left(I_{O,plat+ball} - x^2 m\right)\end{aligned}}$$

5

SIMPLIFY EQ FROM #2 WITH KNOWN α_B

$$\ddot{x} = \ddot{\theta}_y (r_p + r_B) - \dot{\theta}_y^2 x + \left(\frac{g}{\sin\theta_y} - 2\ddot{x}\right) \frac{5}{4 r_B} r_B$$

$$\boxed{\frac{7}{2}\ddot{x} - \ddot{\theta}_y (r_p + r_B) = \frac{5g}{4\sin\theta_y} - \dot{\theta}_y^2 x}$$

SECOND EQUATION ALREADY IN PROPER FORM (FROM #4)

6

COMBINE EQUATIONS FROM #4, #5 INTO MATRIX FORM

$$\begin{array}{cc} \ddot{x} & \ddot{\theta}_y \end{array}$$

$$\begin{bmatrix} \frac{7}{2} & -(r_p + r_B) \\ m r_p & I_{O,plate+ball} - x^2 m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_y \end{bmatrix} =$$

$$\begin{bmatrix} \dfrac{T_x \, r_m \sin(\alpha_x + \theta_x)}{l_p \sin(\alpha_x - \theta_y) - r_p \cos(\alpha_x - \theta_y)} + g\left[\left(r_G M + (r_p + r_B) m\right)\sin\theta_y + \left(\dfrac{r_B}{\sin(2\theta_y)} - x\right) m \cos\theta_y\right] \\ - \dfrac{x m}{r_B}\left[\left(\dot{x} - r_B \dot{\theta}_y\right)^2 + \dot{\theta}_y^2\left[(r_p + r_B)(r_B - 1) - r_B^2\right]\right] \\ \dfrac{5}{4}\dfrac{g}{\sin\theta_y} - \dot{\theta}_y^2 x \end{bmatrix}$$

ASSUMES THESE KNOWNS:

- T_x
- r_m
- α_x ← Note: this can be assumed to be 90° with fair accuracy
- θ_x
- l_p
- r_p
- θ_y, $\dot{\theta}_y$
- g ← gravity
- r_G
- M ← Mass of plate
- r_p
- r_B
- m ← mass of Ball
- x, $\dot{x}$
- $I_{O,plate+ball}$ ← Moment of Inertia of plate and ball about point O
  - * This could be calculated with the known centroidal moment of inertia of the plate and parallel axis theorem

$$I_{O,plate+ball} = \underbrace{\bar{I}_{plate} + M r_G^2}_{plate} + \underbrace{\frac{2}{5} m r_B^2 + m\left(x^2 + (r_p + r_B)^2\right)}_{ball}$$